

## REFLECTION PRINCIPLE FOR VISCOSITY SOLUTIONS OF THE HOMOGENEOUS REAL MONGE–AMPÈRE EQUATION

MIKA KOSKENOJA

**ABSTRACT.** We prove that the reflection principle holds for viscosity solutions of the homogeneous real Monge–Ampère equation. Before stating our main result we consider the reflection of convex functions and viscosity sub- and supersolutions of the real Monge–Ampère equation. Only elementary arguments provided by the concepts of even reflection and viscosity solution are used to obtain the results.

### 1. INTRODUCTION

The real Monge–Ampère equation

$$\det D^2 u = \det \left[ \frac{\partial^2 u}{\partial x_j \partial x_k} \right] = f \quad (1.1)$$

is a second order nonlinear partial differential equation which is not elliptic, in general. An admissible definition of weak type solutions of the Monge–Ampère equation was given by A. D. Aleksandrov, see [15], [10] and [8]. Aleksandrov type weak solutions are called generalized (or Aleksandrov) solutions of the Monge–Ampère equation. The concept of viscosity solutions as weak solutions of first and second order partial differential equations was introduced by Evans [6] and Crandall and Lions [5] in the early 1980’s. A decade later it was thoroughly studied by Crandall, Ishii and Lions [4]. Caffarelli [2] showed that the definitions of generalized and viscosity solutions of the Monge–Ampère equation are equivalent.

Several types of functions and, in particular, solutions of homogeneous partial differential equations can be extended across a flat boundary by applying reflection. Classically, it is used to strong type equations but later on also to some weak type equations. The original reflection principle is due to H. A. Schwarz and it is stated for analytic functions given in the upper half unit disk of the complex plane. Then there is an analytic extension over the real axis to the whole unit disk which is obtained by reflection. A corresponding principle holds for harmonic functions, see [1, Proof of Theorem 1.3.6]. These old results have several much

---

2000 *Mathematics Subject Classification.* 35K96.

*Key words and phrases.* Continuation; extension; reflection principle; real Monge–Ampère equation; Hessian matrix; homogeneous equation; weak solution; viscosity solution.

©2019 Ilirias Research Institute, Prishtinë, Kosovë.

Submitted August 13, 2018. Published April 9, 2019.

Communicated by C. Tunc.

younger generalizations, for example, to quasiregular mappings and elliptic partial differential equations, see e.g. [11].

Before studying the reflection for viscosity solutions of the Monge–Ampère equation (in Section 4) we consider convexity properties of even reflected functions in Section 3. We show that the even reflection preserves convexity of any nonnegative function satisfying a natural boundary condition towards the reflection boundary (Theorem 3.2). This is a key observation in this approach since the set of admissible viscosity solutions of the Monge–Ampère equation consists of convex functions.

## 2. NOTATION AND TERMINOLOGY

Let  $\Omega \subset \mathbb{R}^n$ . A function  $u: \Omega \rightarrow \mathbb{R}$  is *convex* in  $\Omega$  if for any  $x, y \in \Omega$  such that

$$[x, y] := \{tx + (1-t)y : t \in [0, 1]\} \subset \Omega$$

we have

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y) \quad (2.1)$$

for all  $t \in [0, 1]$ . If the inequality (2.1) is strict for all  $t \in (0, 1)$  and for any  $x, y \in \Omega$  such that  $x \neq y$  and  $[x, y] \subset \Omega$ , then the function  $u$  is said to be *strictly convex* in  $\Omega$ . Convex functions  $u: \Omega \rightarrow \mathbb{R}$  are locally Lipschitz and, in particular, continuous whenever  $\Omega$  is open.

Let  $\Omega \subset \mathbb{R}^n$  be open. Suppose that  $u: \Omega \rightarrow \mathbb{R}$  is convex (and hence continuous). If  $x_0 \in \Omega$ , we denote

$$\Phi_{\max}(\Omega, u, x_0) = \{\varphi \in C^2(\Omega) : \varphi \text{ is convex and } x_0 \text{ is a local maximum of } u - \varphi\}$$

and

$$\Phi_{\min}(\Omega, u, x_0) = \{\varphi \in C^2(\Omega) : \varphi \text{ is convex and } x_0 \text{ is a local minimum of } u - \varphi\}$$

which are the sets of test functions in the sequel. Note that if a function  $\varphi \in C^2(\Omega)$  is convex, then the Hesse matrix  $D^2\varphi$  is positive semidefinite. Therefore its all eigenvalues are nonnegative, and hence  $\det D^2\varphi \geq 0$  as the product of the eigenvalues.

Let  $f \geq 0$  in  $\Omega$ . Then  $u$  is a *viscosity subsolution* of the Monge–Ampère equation  $\det D^2u = f$  in  $\Omega$  if

$$\det D^2\varphi(x_0) \geq f(x_0)$$

whenever  $x_0 \in \Omega$  and  $\varphi \in \Phi_{\max}(\Omega, u, x_0)$ . Correspondingly,  $u$  is a *viscosity supersolution* of the Monge–Ampère equation  $\det D^2u = f$  in  $\Omega$  if

$$\det D^2\varphi(x_0) \leq f(x_0)$$

whenever  $x_0 \in \Omega$  and  $\varphi \in \Phi_{\min}(\Omega, u, x_0)$ . In particular,  $u$  is a *viscosity solution* of the Monge–Ampère equation  $\det D^2u = f(x_0)$  in  $\Omega$  if it is both a viscosity subsolution and a viscosity supersolution.

In this paper we consider viscosity solutions of the Monge–Ampère equation  $\det D^2u = c$  where  $c$  is a nonnegative constant, in particular the homogeneous case where  $c = 0$ . It could be possible to study the general equation  $\det D^2u = f$ , but then the function  $f$  should be reflected, in addition to  $u$ , and it is not clear how to do it sensibly.

We next fix our notation regarding the reflection in  $\mathbb{R}^n$ . Let  $G_+$  be a domain in  $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ . Let  $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the *reflection* with respect to  $\partial\mathbb{R}_+^n$ , that is,  $P(x) = P(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n)$ . Suppose that

there exists a non-empty set  $G_0 \subset \partial\mathbb{R}_+^n$  open in  $\partial G_+$ . Write  $G = G_+ \cup G_0 \cup G_-$  where  $G_- = PG_+$ . Then  $G$  is a domain (i.e. an open and connected set) in  $\mathbb{R}^n$ .

Assume that a function  $u: G_+ \rightarrow \mathbb{R}$  fulfills the boundary condition

$$\lim_{\substack{x \rightarrow x_0 \\ x \in G_+}} u(x) = 0 \text{ for all } x_0 \in G_0. \quad (2.2)$$

Then the *even reflected function*  $\hat{u}: G \rightarrow \mathbb{R}$  is

$$\hat{u}(x) = \begin{cases} u(x), & x \in G_+, \\ 0, & x \in G_0, \\ u(P(x)), & x \in G_-. \end{cases} \quad (2.3)$$

The *odd reflected function*  $\tilde{u}: G \rightarrow \mathbb{R}$  is defined correspondingly but it has negative sign in the reflected side of  $G$ , this is,  $\tilde{u}(x) = -u(P(x))$  whenever  $x \in G_-$ .

Reflected functions  $\tilde{u}$  and  $\hat{u}$  preserve continuity of a function  $u$  given in  $G_+$  and satisfying (2.2), and reflection may be used to extend several types of functions and solutions of certain equations across a flat boundary. Concerning the extension of classical solutions of the homogeneous real Monge–Ampère equation, see [13].

An inspiration to this study is the *reflection principle for subharmonic functions*. Suppose that  $u: G_+ \rightarrow \mathbb{R}$  is a nonnegative and subharmonic function which satisfies the boundary condition (2.2). First we note that  $\hat{u}$  is upper semicontinuous in  $G$ , since it is upper semicontinuous in  $G_+ \cup G_-$  and continuous across the reflection boundary  $G_0$ . It is easy to see that  $\hat{u}$  satisfies the submean value property in  $G$ . For example, if  $x_0 \in G_0$  and  $r > 0$  is such that  $B(x_0, r) \subset G$ , then

$$\frac{1}{m(B(x_0, r))} \int_{B(x_0, r)} \hat{u}(x) dx \geq 0 = \hat{u}(x_0)$$

since  $\hat{u}(x) \geq 0$  for all  $x \in G$ . It follows that  $\hat{u}$  is subharmonic in  $G$ .

Finally in this preparatory section we introduce a generalized notion of the even reflection. Suppose that  $\varphi: G_+ \cup G_0 \cup G_- = G \rightarrow \mathbb{R}$ . It is not assumed here that  $\varphi \equiv 0$  in  $G_0$ . Then the *mutually even reflected function*  $\hat{\varphi}: G \rightarrow \mathbb{R}$  is given by

$$\hat{\varphi}(x) = \varphi(P(x)), \quad x \in G. \quad (2.4)$$

Note that if  $\varphi: G \rightarrow \mathbb{R}$ , then  $\hat{\varphi}(x) = \varphi(x)$  for every  $x \in G$ . Respectively, we may consider mutually odd reflected functions, but we do not need them in this study.

### 3. CONVEXITY OF EVEN REFLECTED FUNCTIONS

**Lemma 3.1.** *If  $x, y \in \mathbb{R}^n$ , then*

$$P(tx + (1-t)y) = tP(x) + (1-t)P(y) \quad (3.1)$$

for all  $t \in [0, 1]$ .

*Proof.* Let  $x, y \in \mathbb{R}^n$ . We have for each  $t \in [0, 1]$  that

$$\begin{aligned} P(tx + (1-t)y) &= P(tx_1 + (1-t)y_1, \dots, tx_{n-1} + (1-t)y_{n-1}, tx_n + (1-t)y_n) \\ &= (tx_1 + (1-t)y_1, \dots, tx_{n-1} + (1-t)y_{n-1}, -tx_n - (1-t)y_n) \\ &= t(x_1, \dots, x_{n-1}, -x_n) + (1-t)(y_1, \dots, y_{n-1}, -y_n) \\ &= tP(x) + (1-t)P(y) \end{aligned} \quad \square$$

**Theorem 3.2.** *Let  $u: G_+ \rightarrow \mathbb{R}$  be nonnegative and convex in  $G_+$  and satisfy the boundary condition (2.2). Then  $\hat{u}: G \rightarrow \mathbb{R}$  is nonnegative and convex in  $G$ . In particular, if  $u$  is strictly convex in  $G_+$ , then  $\hat{u}$  is strictly convex in  $G$ .*

*Proof.* It is clear by the definition that the even reflected function  $\hat{u}$  is nonnegative in  $G$ . Let  $x, y \in G$  be such that  $[x, y] \subset G$ . We separate our proof to four nonoverlapping cases, of which the first one is obviously true.

Case 1: Both points  $x, y$  are in  $G_+$ . Then  $\hat{u}(tx + (1-t)y) \leq t\hat{u}(x) + (1-t)\hat{u}(y)$  for every  $t \in [0, 1]$  since  $\hat{u} \equiv u$  in  $G_+$  and  $u$  is convex in  $G_+$ .

Case 2: Both points  $x, y$  are in  $G_-$ . Since  $G_- = P(G_+) \subset \mathbb{R}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n: x_n < 0\}$  and  $\mathbb{R}^n$  is convex,  $z_t = tx + (1-t)y$  can not lie in  $G_0 \cup G_+ \subset \mathbb{R}^n \setminus \mathbb{R}^n_-$  for any  $t \in [0, 1]$ . Hence  $z_t = tx + (1-t)y \in G_-$  for each  $t \in [0, 1]$ , and therefore

$$P([x, y]) = \{P(z_t) = P(tx + (1-t)y): 0 \leq t \leq 1\} \subset G_+.$$

Since  $u$  is convex in  $G_+$ , formula (3.1) yields now

$$\begin{aligned} \hat{u}(tx + (1-t)y) &= u(P(tx + (1-t)y)) = u(tP(x) + (1-t)P(y)) \\ &\leq tu(P(x)) + (1-t)u(P(y)) = t\hat{u}(x) + (1-t)\hat{u}(y) \end{aligned}$$

for every  $t \in [0, 1]$ .

Case 3: One of the points  $x, y$  is in  $G_0$ . Assume that  $x = (x_1, \dots, x_{n-1}, 0) \in G_0$ . If also  $y \in G_0$ , then we may write  $y = (y_1, \dots, y_{n-1}, 0)$  and we have

$$\begin{aligned} z_t &= tx + (1-t)y = t(x_1, \dots, x_{n-1}, 0) + (1-t)(y_1, \dots, y_{n-1}, 0) \\ &= (tx_1 + (1-t)y_1, \dots, tx_{n-1} + (1-t)y_{n-1}, 0) \in G_0 \end{aligned}$$

for all  $t \in [0, 1]$ . This yields

$$\hat{u}(tx + (1-t)y) = 0 = 0 + 0 = t\hat{u}(x) + (1-t)\hat{u}(y)$$

for every  $t \in [0, 1]$ . Otherwise, if  $y \notin G_0$ , then  $y \in G_+$  or  $y \in G_-$ . It follows that for  $t = 1$  we have  $z_t = tx + (1-t)y \in G_0$  and for all  $t \in [0, 1]$   $z_t \in G_+$  if  $y \in G_+$  and  $z_t \in G_-$  if  $y \in G_-$ . Then by earlier Cases 1 and 2 and by continuity of  $\hat{u}$  in  $G$  we have

$$\begin{aligned} \hat{u}(tx + (1-t)y) &= \lim_{\substack{\xi \rightarrow x \\ \xi \in [x, y]}} \hat{u}(t\xi + (1-t)y) \leq \lim_{\substack{\xi \rightarrow x \\ \xi \in [x, y]}} t\hat{u}(\xi) + (1-t)\hat{u}(y) \\ &= t\hat{u}(x) + (1-t)\hat{u}(y) \end{aligned}$$

for every  $t \in [0, 1]$ .

Case 4: One of the points  $x, y$  is in  $G_+$  and the other is in  $G_-$ . We may suppose that  $x \in G_+$  and  $y \in G_-$ . Note that now  $x \neq y$  and hence  $|x - y| > 0$ . Then there is a unique  $t_0 \in (0, 1)$  such that  $z_0 := t_0x + (1-t_0)y \in G_0$ . Choose now an arbitrary  $t \in [0, 1]$ .

Subcase 4.1: If  $t = 0$  or  $t = 1$ , then  $\hat{u}(tx + (1-t)y) = t\hat{u}(x) + (1-t)\hat{u}(y)$ .

Subcase 4.2: If  $t = t_0$ , then  $tx + (1-t)y = t_0x + (1-t_0)y = z_0 \in G_0$ , and we get

$$\hat{u}(tx + (1-t)y) = \hat{u}(z_0) = 0 \leq t\hat{u}(x) + (1-t)\hat{u}(y),$$

because  $\hat{u}$  is nonnegative in  $G$ .

Subcase 4.3: If  $t \in (0, t_0)$ , then  $tx + (1-t)y \in G_-$ . There is  $t_1 \in (0, 1)$  such that  $tx + (1-t)y = t_1z_0 + (1-t_1)y$ . It follows that  $t(x - y) = t_1(z_0 - y)$  and hence

$$\frac{t_1}{t} = \frac{|x - y|}{|z_0 - y|} > 1 \quad \text{as } |x - y| > |z_0 - y|.$$

This implies that  $t_1 > t$  and therefore by Case 3 we have

$$\begin{aligned}\hat{u}(tx + (1-t)y) &= \hat{u}(t_1 z_0 + (1-t_1)y) \leq t_1 \hat{u}(z_0) + (1-t_1) \hat{u}(y) \\ &= (1-t_1) \hat{u}(y) \leq (1-t) \hat{u}(y) \leq t \hat{u}(x) + (1-t) \hat{u}(y).\end{aligned}$$

Subcase 4.4: If  $t \in (t_0, 1)$ , then  $tx + (1-t)y \in G_+$ . There is  $t_2 \in (0, 1)$  such that  $tx + (1-t)y = t_2 x + (1-t_2)z_0$ . It follows that

$$\begin{aligned}t(x-y) &= z_0 - y + t_2(x-z_0) + (1-t_2)(x-z_0) - (1-t_2)(x-z_0) \\ &= z_0 - y + x - z_0 - (1-t_2)(x-z_0) = (x-y) - (1-t_2)(x-z_0)\end{aligned}$$

and hence  $(1-t)(x-y) = (1-t_2)(x-z_0)$  which yields

$$\frac{1-t_2}{1-t} = \frac{|x-y|}{|x-z_0|} > 1 \quad \text{as } |x-y| > |x-z_0|.$$

This implies that  $t_2 < t$  and therefore by Case 3 we have

$$\begin{aligned}\hat{u}(tx + (1-t)y) &= \hat{u}(t_2 x + (1-t_2)z_0) \leq t_2 \hat{u}(x) + (1-t_2) \hat{u}(z_0) \\ &= t_2 \hat{u}(x) \leq t \hat{u}(x) \leq t \hat{u}(x) + (1-t) \hat{u}(y).\end{aligned}$$

Consequently, by Cases 1–4

$$\hat{u}(tx + (1-t)y) \leq t \hat{u}(x) + (1-t) \hat{u}(y)$$

for every  $t \in [0, 1]$ . This means that  $\hat{u}$  is convex in  $G$ .

The additional part of the lemma is obtained since all previous convexity inequalities are strict if  $u$  is strictly convex in  $G_+$  and  $x \neq y$ ,  $t \in (0, 1)$ .  $\square$

Moreover, we have the following properties of mutually even reflected functions.

**Lemma 3.3.** *Let  $\varphi: G \rightarrow \mathbb{R}$ .*

(i) *If  $\varphi \in C^2(G)$ , then  $\hat{\varphi} \in C^2(G)$  and*

$$\det D^2 \hat{\varphi}(x) = \det D^2 \varphi(P(x)) \quad \text{for all } x \in G.$$

(ii) *If  $\varphi$  is (strictly) convex in  $G$ , then  $\hat{\varphi}$  is (strictly) convex in  $G$ .*

*Proof.* (i) See [13, Lemma 3.6 and Theorem 3.12].

(ii) Let  $\varphi$  be convex in  $G$ . Suppose that  $x, y \in G$  are such that  $[x, y] \subset G$ . It follows from Lemma 3.1 that

$$\begin{aligned}I(P(x), P(y)) &= \{tP(x) + (1-t)P(y) : 0 \leq t \leq 1\} \\ &= \{P(tx + (1-t)y) : 0 \leq t \leq 1\} \subset P(P(G)) = G.\end{aligned}$$

Since  $\varphi$  is convex in  $G$ , formulas (2.4) and (3.1) yield now

$$\begin{aligned}\hat{\varphi}(tx + (1-t)y) &= \varphi(P(tx + (1-t)y)) = \varphi(tP(x) + (1-t)P(y)) \\ &\leq t\varphi(P(x)) + (1-t)\varphi(P(y)) = t\hat{\varphi}(x) + (1-t)\hat{\varphi}(y)\end{aligned} \tag{3.2}$$

for every  $t \in [0, 1]$ . The strict convexity part of the result follows now immediately, since the inequality in (3.2) is strict if  $\varphi$  is strictly convex in  $G$  and  $x \neq y$ ,  $t \in (0, 1)$ .  $\square$

## 4. REFLECTION PRINCIPLE FOR VISCOSITY SOLUTIONS

We first prove that the even reflection preserves viscosity supersolutions of the Monge–Ampère equation  $\det D^2u = c$  whenever  $c \geq 0$  is a constant.

**Theorem 4.1.** *Let  $u: G_+ \rightarrow \mathbb{R}$  be nonnegative and convex in  $G_+$  and satisfy the boundary condition (2.2). If  $u$  is a viscosity supersolution of the Monge–Ampère equation  $\det D^2u = c$  in  $G_+$  where  $c \geq 0$ , then  $\hat{u}$  is a viscosity supersolution of the Monge–Ampère equation  $\det D^2\hat{u} = c$  in  $G$ .*

*Proof.* Let  $c \geq 0$  and  $u$  be a viscosity supersolution of the equation  $\det D^2u = c$  in  $G_+$ . By the definition,  $u$  is convex in  $G_+$ . Now Theorem 3.2 tells us that  $\hat{u}$  is convex in  $G$  and hence it is an admissible viscosity supersolution in  $G$ .

Suppose first that  $x_0 \in G_+$  and that  $\varphi \in \Phi_{\min}(G, \hat{u}, x_0)$ . Then there is  $r > 0$  such that  $B(x_0, r) \subset G_+$  and  $(\hat{u} - \varphi)(x) \geq (\hat{u} - \varphi)(x_0)$  for all  $x \in B(x_0, r)$ . Because  $\hat{u} \equiv u$  in  $G_+$ , we have  $(u - \varphi|_{G_+})(x) \geq (u - \varphi|_{G_+})(x_0)$  for all  $x \in B(x_0, r)$  and hence  $\varphi|_{G_+} \in \Phi_{\min}(G_+, u, x_0)$ . Since  $u$  is a viscosity supersolution of the equation  $\det D^2u = c$  in  $G_+$ , it follows by the definition that  $\det D^2\varphi(x_0) = \det D^2\varphi|_{G_+}(x_0) \leq c$ .

Suppose then that  $x_0 \in G_-$  and that  $\varphi \in \Phi_{\min}(G, \hat{u}, x_0)$ . Then there is  $r > 0$  such that  $B(x_0, r) \subset G_-$  and  $(\hat{u} - \varphi)(x) \geq (\hat{u} - \varphi)(x_0)$  for all  $x \in B(x_0, r)$ . Now  $P(B(x_0, r)) = B(P(x_0), r) \subset G_+$ ,  $\hat{\varphi} \in C^2(G)$  and

$$(u - \hat{\varphi})(y) = (\hat{u} - \varphi)(P(y)) \geq (\hat{u} - \varphi)(x_0) = (u - \hat{\varphi})(P(x_0))$$

for all  $y \in B(P(x_0), r)$  since every such  $y$  satisfies  $P(y) \in B(x_0, r)$ . Therefore  $(u - \hat{\varphi}|_{G_+})(y) \geq (u - \hat{\varphi}|_{G_+})(P(x_0))$  for all  $y \in B(P(x_0), r)$  and hence  $\hat{\varphi}|_{G_+} \in \Phi_{\min}(G_+, u, P(x_0))$ . It follows by Lemma 3.3(i) that

$$\det D^2\varphi(x_0) = \det D^2\hat{\varphi}(P(x_0)) = \det D^2\hat{\varphi}|_{G_+}(P(x_0)) \leq c$$

since  $u$  is a viscosity supersolution of the equation  $\det D^2u = c$  in  $G_+$  and  $P(x_0) \in G_+$ .

Finally, let  $x_0 \in G_0$ . Suppose that  $\varphi \in \Phi_{\min}(G, \hat{u}, x_0)$ . Then there is  $r > 0$  such that  $B(x_0, r) \subset G$  and that

$$(\hat{u} - \varphi)(x) \geq (\hat{u} - \varphi)(x_0) = -\varphi(x_0)$$

for all  $x \in B(x_0, r)$ . It follows that  $\varphi(x) \leq \varphi(x_0)$  for all  $x \in B(x_0, r) \cap G_0$  since  $\hat{u} = 0$  in  $G_0$ . This implies  $\varphi(x) = \varphi(x_0)$  for all  $x \in B(x_0, r) \cap G_0$  since  $\varphi$  is convex. Therefore  $\varphi$  is constant in  $B(x_0, r) \cap G_0$  which gives  $\det D^2\varphi(x_0) = 0 \leq c$ . All in all,  $\hat{u}$  is a viscosity supersolution of the equation  $\det D^2\hat{u} = c$  in  $G$ .  $\square$

Note that if  $x_0 \in G_0$ , then  $\varphi \equiv 0$  belongs to  $\Phi_{\min}(G, \hat{u}, x_0)$ . Hence the set of test functions  $\Phi_{\min}(G, \hat{u}, x_0)$  is always nonempty. On the contrary it is possible that the set  $\Phi_{\max}(G, \hat{u}, x_0)$  is empty which can be seen as follows.

**Example 4.2.** Let  $B_+ = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1 \text{ and } y > 0\}$  be the upper half unit disk. Given  $B_0 = \{(x, y) \in \mathbb{R}^2: -1 < x < 1 \text{ and } y = 0\}$  the set  $B = B_+ \cup B_0 \cup B_-$  is the open unit disk and  $B_- = PB_+$ . Suppose that  $u: B_+ \rightarrow \mathbb{R}$  is given by  $u(x, y) = y$  where  $(x, y) \in B_+$ . Then  $u$  is convex in  $B_+$  and it fulfills the equation  $\det D^2u = 0$  strongly in  $B_+$ . Moreover,  $u$  extends continuously to zero in  $B_0$  and  $\hat{u}(x, y) = -y$  if  $(x, y) \in B_-$ . It follows that the even reflected function  $\hat{u}$  is convex in  $B$ . Now

$$\frac{\partial \hat{u}}{\partial y}(x, y) = \begin{cases} 1, & (x, y) \in B_+, \\ -1 & (x, y) \in B_-, \end{cases}$$

and hence at every  $(x, 0) \in B_0$  the directional derivative of  $\hat{u}$  to the direction  $(x, 1)$  is equal to 1 and to the opposite direction  $(x, -1)$  it is equal to  $-1$ . Therefore there cannot be a differentiable function  $\varphi$  touching  $\hat{u}$  from above at  $(x, 0) \in B_0$ . Hence  $\Phi_{\max}(G, \hat{u}, x_0) = \emptyset$  and the condition for viscosity subsolutions is trivially verified. Moreover, by Theorem 4.1  $\hat{u}$  is a viscosity supersolution of the equation  $\det D^2 \hat{u} = 0$  in  $B$ , and hence  $\hat{u}$  is a viscosity solution in  $B$ . But  $\hat{u}$  is not a solution of the equation  $\det D^2 \hat{u} = 0$  in the classical sense, since it is not differentiable at any point of  $B_0$ .

Suppose that  $f: \Omega \rightarrow \mathbb{R}$ ,  $x \in \Omega$  and  $v \in \mathbb{R}^n$ . The *one-sided directional derivatives* of  $f$  at  $x$  with respect to  $v$  are defined by

$$\partial_v^+ f(x) = \lim_{t \rightarrow 0^+} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad \partial_v^- f(x) = \lim_{t \rightarrow 0^-} \frac{f(x + tv) - f(x)}{t}.$$

If both one-sided directional derivatives  $\partial_v^+ f(x)$  and  $\partial_v^- f(x)$  exist and they are equal, then their common value is called the *(two-sided) directional derivative* of  $f$  at  $x$  with respect to  $v$  and it is denoted by  $\partial_v f(x)$ . Note that by the definition  $\partial_v^+ f(x) = -\partial_{-v}^- f(x)$  whenever exist. If  $f$  is convex in  $\Omega$ , then one-sided directional derivatives of  $f$  exist at every point  $x \in \Omega$  with respect to any  $v$  and  $\partial_v^+ f(x) \geq \partial_v^- f(x)$ . Moreover, existence of all partial derivatives of a convex function  $f$  at  $x$  ensures differentiability of  $f$  at  $x$  which is, of course, not true in general. For comprehensive discussions of convex functions and directional derivatives, see the monographs [14], [16] and [17].

**Lemma 4.3.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $f, g: \Omega \rightarrow \mathbb{R}$  be nonnegative functions such that  $f(x_0) = g(x_0) = 0$  and  $f(x) \leq g(x)$  in a neighbourhood  $U$  of  $x_0 \in \Omega$ . If  $g$  is differentiable at  $x_0$ , then  $f$  is differentiable at  $x_0$  and  $\partial_v f(x_0) = \partial_v g(x_0) = 0$  for every  $v \in \mathbb{R}^n$ .*

*Proof.* Suppose that  $g$  is differentiable at  $x_0 \in \Omega$ . Let  $v \in \mathbb{R}^n$ . It follows that the directional derivative  $\partial_v g(x_0)$  exists. Nonnegativity of  $g$  gives

$$\partial_v^+ g(x) = \lim_{t \rightarrow 0^+} \frac{g(x_0 + tv) - g(x_0)}{t} = \lim_{t \rightarrow 0^+} \frac{g(x_0 + tv)}{t} \geq 0$$

and

$$\partial_v^- g(x) = \lim_{t \rightarrow 0^-} \frac{g(x_0 + tv) - g(x_0)}{t} = \lim_{t \rightarrow 0^-} \frac{g(x_0 + tv)}{t} \leq 0$$

which implies  $\partial_v g(x_0) = 0$ . Now, let  $\varepsilon > 0$ . Since  $f(x_0) = 0$  and  $0 \leq f(x) \leq g(x)$  in a neighbourhood of  $x_0$ , there is a real number  $t_\varepsilon > 0$  such that

$$\left| \frac{f(x_0 + tv) - f(x_0)}{t} \right| = \left| \frac{f(x_0 + tv)}{t} \right| \leq \left| \frac{g(x_0 + tv)}{t} \right| < \varepsilon$$

for every  $t \in (-t_\varepsilon, t_\varepsilon)$ . Hence  $\partial_v f(x_0)$  exists and is equal to zero.  $\square$

Next we prove that the even reflection preserves viscosity subsolutions of the homogeneous Monge–Ampère equation  $\det D^2 u = 0$ . This observation follows easily from the definition of subsolutions, since  $\det D^2 \varphi$  is nonnegative for all convex twice continuously differentiable functions  $\varphi$ . In general, we can not hope a better result, for example, a similar reflection property for the nonhomogeneous Monge–Ampère equation  $\det D^2 u = c$  where  $c > 0$ , since typically  $\det D^2 \varphi(x_0) = 0$  at the points  $x_0 \in G_0$  whenever  $\varphi$  is convex and twice continuously differentiable.

**Theorem 4.4.** *Let  $u: G_+ \rightarrow \mathbb{R}$  be nonnegative and convex in  $G_+$  and satisfy the boundary condition (2.2). If  $u$  is a viscosity subsolution of the homogeneous Monge–Ampère equation  $\det D^2u = 0$  in  $G_+$ , then  $\hat{u}$  is a viscosity subsolution of the homogeneous Monge–Ampère equation  $\det D^2\hat{u} = 0$  in  $G$ .*

*Proof.* Let  $u$  be a viscosity subsolution of the equation  $\det D^2u = 0$  in  $G_+$ . Again, we see that  $\hat{u}$  is an admissible viscosity subsolution in  $G$ . Moreover, analogously as in the proof of Theorem 4.1, at the points  $x_0$  in  $G_+$  and  $G_-$  we have  $\det D^2\varphi(x_0) \geq 0$  for every test function  $\varphi \in \Phi_{\max}(G, \hat{u}, x_0)$ .

Now, let  $x_0 \in G_0$ . Suppose first that  $\frac{\partial \hat{u}}{\partial x_n}(x_0)$  does not exist or is not equal to zero if exists. Then by Lemma 4.3 the set of test functions  $\Phi_{\max}(G, \hat{u}, x_0)$  is empty and the condition for viscosity subsolutions is trivially verified.

Suppose then that  $\frac{\partial \hat{u}}{\partial x_n}(x_0) = 0$ . Moreover, suppose that there exists a test function  $\varphi \in \Phi_{\max}(G, \hat{u}, x_0)$ , otherwise the condition for viscosity subsolutions is trivially verified again. We may assume that  $\varphi(x_0) = \hat{u}(x_0) = 0$ . Since  $\varphi$  is convex, it is positive semidefinite, and therefore all eigenvalues of  $D^2\varphi(x_0)$  are nonnegative. Because the determinant of a matrix is the product of its eigenvalues, we have

$$\det D^2\varphi(x_0) \geq 0 = \varphi(x_0).$$

Now we conclude that  $\hat{u}$  is a viscosity subsolution in  $G$ .  $\square$

**Corollary 4.5.** *Let  $u: G_+ \rightarrow \mathbb{R}$  be nonnegative and convex in  $G_+$  and satisfy the boundary condition (2.2). If  $u$  is a viscosity solution of the homogeneous Monge–Ampère equation  $\det D^2u = 0$  in  $G_+$ , then  $\hat{u}$  is a viscosity solution of the homogeneous Monge–Ampère equation  $\det D^2\hat{u} = 0$  in  $G$ .*

*Proof.* The result is an immediate consequence of Theorems 4.1 and 4.4. In Theorem 4.1 we choose  $c = 0$ .  $\square$

**Conclusion.** We have proved that the even reflection preserves viscosity subsolutions, supersolutions and solutions of the homogeneous real Monge–Ampère equation  $\det D^2\hat{u} = 0$ . For viscosity supersolutions the reflection principle is valid also for the nonhomogeneous equation  $\det D^2\hat{u} = c$ , where  $c > 0$ , but obviously in that case the reflection principle does not hold for viscosity subsolutions and solutions.

It is possible to apply the concepts and methods used in this paper to obtain reflection principles for viscosity solutions of some general type second order partial differential equations considered in [4]. In particular, if the sets of test functions and admissible viscosity solutions are included in the class of convex functions, then the considerations of this paper are applicable. Moreover, it is easy to see that if a function  $u$  is upper semicontinuous in  $G_+$  and satisfies the condition (2.2), then  $\hat{u}$  is upper semicontinuous in  $G$ . This gives an analogous result for Theorem 3.2 when the set of admissible viscosity solutions consists of nonnegative upper semicontinuous functions.

**Acknowledgments.** The author would like to thank the anonymous referee for his/her comments that helped to improve this article.

#### REFERENCES

- [1] D.H. Armitage, S.J. Gardiner, *Classical Potential Theory*, Springer Monogr. Math., Springer-Verlag, London, 2001.
- [2] L. A. Caffarelli, *Interior  $W^{1,p}$  estimates for solutions of the Monge–Ampère equation*, Ann. of Math. (2) **131** (1990) 135–150.



- [3] M. G. Crandall, *Viscosity solutions: a primer*, in: Viscosity solutions and applications (Montecatini Terme, 1995), pp. 1–43, Lecture Notes in Math. **1660**, Springer, Berlin, 1997.
- [4] M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992) 1–67.
- [5] M. G. Crandall, P.-L. Lions, *Viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983) 1–42.
- [6] L. C. Evans, *On solving certain nonlinear partial differential equations by accretive operator methods*, Israel J. Math. **36** (1980) 225–247.
- [7] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
- [8] A. Figalli, *The Monge–Ampère Equation and Its Applications*, EMS Zurich Lectures in Advanced Mathematics **22**, Eur. Math. Soc., 2017.
- [9] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order. Second Edition*, Grundlehren Math. Wiss. **224**, Springer-Verlag, Berlin, 1983.
- [10] C. E. Gutiérrez, *The Monge–Ampère Equation*, Progress in Nonlinear Differential Equations and Their Applications **44**, Birkhäuser, Boston, 2001.
- [11] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monogr., Oxford Univ. Press, Oxford, 1993.
- [12] F. John, *Continuation and reflection of solutions of partial differential equations*, Bull. Amer. Math. Soc. **63** (1957) 327–344.
- [13] M. Koskenoja, *Reflection principle for classical solutions of the homogeneous real Monge–Ampère equation*, Cogent Math. **2** (2015) 1024993.
- [14] C. P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics, Springer, New York, 2006.
- [15] A. V. Pogorelov, *Monge–Ampère Equations of Elliptic Type*, Noordhoff, Groningen, 1964.
- [16] A. W. Roberts, D. E. Varberg, *Convex Functions*, Academic Press, New York and London, 1973.
- [17] R. R. Rockafellar, *Convex Analysis*. Princeton Math. Ser. **28**, Princeton University Press, Princeton, New Jersey, 1970.

MIKA KOSKENOJA

DEPARTMENT OF MATHEMATICS AND STATISTICS, P.O. BOX 68 (PIETARI KALMIN KATU 5), FI-00014 UNIVERSITY OF HELSINKI, FINLAND

*E-mail address:* mika.koskenoja@helsinki.fi